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Connected Graphs with Unlabeled End-Points

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ABSTRACT

A formula is derived for the number of connected graphs in which all points except end-points are labeled.

An earlier version of the preceding note by Harary, Mowshowitz, and Riordan [2], raised the problem of determining the number of graphs in which only the points that are not end-points are labeled. In this note we derive a formula for $E(t+k, e+k, k)$, the number of connected graphs G with $t+k$ points and $e+k$ edges such that $t(\geqslant 3)$ of the points are labeled and are not end-points, and k of the points are not labeled and are end-points.

If the k unlabeled end-points are removed from G, then the resulting graph G' is connected and has t labeled points and e edges; suppose G' has exactly i end-points, where $0 \leqslant i \leqslant t-1$. The number $H(t, e, i)$ of such graphs G' is given by the formula

$$H(t, e, i) = \sum_{j=i}^{t-1} (-1)^{j-i} \binom{j}{i}\binom{t}{j} C(t-j, e-j)(t-j)^j, \tag{1}$$

where $C(n, m)$ denotes the number of connected graphs with n points and m edges. (Gilbert [1] gave a generating function for the numbers $C(n, m)$.)

Formula (1) follows immediately from the method of inclusion and exclusion upon observing that no two end-points of a connected graph can be joined to each other if $t > 2$. (We remark that a special case of (1) was used in [3] to derive Cayley's formula for the number of trees and Rényi's formula for the number of trees with a given number of end-points.)

There are

$$(-1)^{k-i}\binom{-t}{k-i} = \binom{k+t-1-i}{t-1}$$

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ways of adding k unlabeled points to the graph G' so that these k points are the only end-points in the resulting graph G (see [3, §2]). Therefore,

$$\begin{aligned} E(t+k, e+k, k) &= \sum_{i=0}^{t-1} (-1)^{k-i} \binom{-t}{k-i} H(t, e, i) \\ &= \sum_{j=1}^{t-1} (-1)^{j+k} \binom{t}{j} C(t-j, e-j)(t-j)^j \sum_{i=0}^{j} \binom{-t}{k-i}\binom{j}{i} \\ &= \sum_{j=0}^{t-1} (-1)^j \binom{t}{j}\binom{k+t-1-j}{k} C(t-j, e-j)(t-j)^j, \end{aligned} \tag{2}$$

by (1). (Formula (2) is also valid when $t = 1$ and 2 if $k \geqslant 2$; notice that $E(t, e, 0) = H(t, e, 0)$.)

If we wish to count the number $T^*(n)$ of trees with $n(>2)$ points in which all points are labeled except the end-points, we set $t = n - k$, and $C(t-j, e-j) = (n-k-j)^{n-k-j-2}$ in (2) and sum from $k = 2$ to $k = n - 1$. Hence,

$$T^*(n) = \sum_{k=2}^{n-1} \sum_{j=0}^{n-k-1} (-1)^j \binom{n-k}{j}\binom{n-1-j}{k} (n-k-j)^{n-k-2}. \tag{3}$$

We leave it for the reader to check that formula (3) agrees with the formulas (1) and (2) in [2].

References

1. E. N. Gilbert, Enumeration of Labeled Graphs, *Canad. J. Math.* **8** (1956), 405–411.
2. F. Harary, A. Mowshowitz, and J. Riordan, Labeled Trees with Unlabeled End-Points, *J. Combinatorial Theory* **6** (1969), 60–64.
3. J. W. Moon, Another Proof of Cayley's Formula for Counting Trees, *Amer. Math. Monthly* **70** (1963), 846–847.